Introduction to Actuarial Science

3A DDEFI M2 AMSE M2 IMSA

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Introduction (1)

- ▷ Specificity of insurance business: inverted production cycle
  Insurance contract = promise
  - ⇒ Importance of **forecast**
  - ⇒ Importance of **regulation**
- ▷ Need to evaluate **ex-ante** and **precisely** the prices (and the risks). That is
  - ▶ To evaluate the (price of) **time** (actualization, link w/ finance)
  - ▶ To evaluate the **risks** (link w/ probability)

That is what ACTUARIAL SCIENCE does

Introduction (2)

Need to differentiate

▷ Life **insurance**

- insurance in case of life or in case of death
- long term
- less hazard

▷ Non-life **insurance**

- IARD (Incendie Accident et Risques Divers) in French
- short term
- high hazard

Outline of the course

- **Life insurance model**
  - Mortality risk and pricing errors
  - Main insurance products: fair premiums and prudent pricing
  - Actuarial Present Value and Notations
  - Exercises
- **Non-life specificity**
  - Provisioning
  - The variability of non-life risks
  - The role of financial markets

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Life insurance

- Insurance in case of life; in case of death
- Long term: pricing of time is important
  - value of 1€ latter?
  - actualization (NPV), what rate?
- Random events
  - use of probability: **"Actuarial Present Value"**
  - what probability(ies)?
- **Pricing** based on **forecasts** of:
  - interest rates
  - mortality rates

A simple example: endowment policy

▷ Commitment: pay the policyholder c€ in k years if she's alive

▷ in French: "capital différé en cas de vie"

▷ Assume an insurer selling n_a such contracts at a premium Π''

▷ Its net profit at the end of the contract (in k years) will be:

$$R_{n_a} = n_a.\Pi''.(1+i)^k - c.\mathcal{N}_V$$

where i is the interest rate
and $\mathcal{N}_V$ represent the # of policyholders alive at $t = k$ (random at $t = 0$)

- assuming that all the policyholders have the same probability p to be alive at $t = k$
- and that all these probabilities are independent, one has

$$\mathbb{E}(R_{n_a}) = n_a.\Pi''.(1+i)^k - c.n_a.p$$
$$\sigma(R_{n_a}) = c.\sigma(R_{n_a}) = c.\sqrt{n_a.p.(1-p)}$$

- Numerical ex.: $n_a = 10,000$; $c = 100,000$; $t = 8$; $i = 6\%$; $p = 0.9865$

and $\Pi'' = 63,000$ give

$$\mathbb{E}(R_{n_a}) = 17,614,290 \qquad \sigma(R_{n_a}) = 1,154,030$$

- Remarks :
  - small standard error; relatively "safe" contract for the insurer
  - here Π'' fixed; in general, look for the premium s.t. $\mathbb{E}(R_1) = 0$ labeled "**actuarially fair premium**"
    "**fair**" : insurer's commitment = insured's commitment
  - the difference between commercial and actuarial premium constitutes the **mathematical reserves** ("provisions mathématiques")

Life tables (1)

- ▷ in previous ex., same survival proba p for all
- ▷ in reality: use of **life tables**
- ▷ that only depend on **age**
- ▷ use of survival probabilities:
  - ▶ if $l_x = (\#$ of ind. aged x at $t = 0)$
  - ▶ and $l_{x+k} = (\#$ of ind. aged x at $t = 0$ alive at $t = k)$
- ▷ $\mathbb{P}(\text{an ind. aged } x \text{ at } t = 0 \text{ is alive at en } t = k) = \frac{l_{x+k}}{l_x}$
- ▷ $\mathbb{P}(\text{an ind. aged } x \text{ at } t = 0 \text{ dies } \textbf{before } t = k) = 1 - \frac{l_{x+k}}{l_x} = \frac{l_x - l_{x+k}}{l_x}$
- ▷ Ex.: $\mathbb{P}(\text{an ind. aged 35 dies before 45}) = 1 - \frac{l_{45}}{l_{35}}$

  $\mathbb{P}(\text{an ind. aged 35 dies between 40 and 45}) = \frac{\dots}{\dots}$

Life tables (2)

- ▷ Survival law of an ind aged x: l_x, l_{x+1},...,l_{x+k},...,l_w where w is the extreme age of life ($\approx$ 110 y.o.)
- ▷ **Life table**: survival law starting from $l_0 = 100,000$

French case: $\exists$ several tables. **Selection established by regulation**

- ▷ TD & TV 88-90 (bylaw of April '93); observations by INSEE 1988-1990
  - ▶ TD 88-90: on a pop. of **males** ; used for insurance in case of **death**
  - ▶ TV 88-90: on a pop. of **females** ; used for insurance in case of **live**
- ▷ replaced by TH and TF 00-02; applicable since 2006
  smoothed : age correction $\leftarrow$ mortality spread between generation
- ▷ HERE we will use TD and TV 88-90 (simpler)

Mortality risk and pricing mistakes

- ▷ Pricing (forecast) and
- ▷ insurer's profit (realization), therefore high depend on:
  - ▶ **assumptions** on mortality (via tables)
  - ▶ and on interest rate(s)
- ▷ Put another way, (life) a insurer faces:
  - ▶ mortality risk
  - ▶ pricing ("of time") mistakes
- ▷ Depending on the product (contract) characteristics
- ▷ these **risks** are more or less significant

The main (simple) products

For the main (simple) products, we will:

- ▷ Compute the fair premium, i.e. the **Actuarial Present Value**
- ▷ Analyze how it depends on (mortality and i.r.) assumptions
- ▷ Define the **prudent pricing**
- ▷ Compute the **variance** of the annual cost (for the insurer)

Endowment policy (is back)

- Recall: pay c€ in k years if alive
- Look for Π s.t. $\mathbb{E}(R_1) = 0$, i.e.

$$\Pi(1+i)^k - c.p = 0$$

where p is the prob that the policyholder will be alive in k year

- Defining $v \equiv \frac{1}{1+i}$ the actualization rate:

$$\Pi = \frac{c.p}{(1+i)^k} = c.v^k.\frac{l_{x+k}}{l_x}$$

- This is the **Actuarial Present Value** of the product / contract

▷ Numerical ex.: $x = 40$; $k = 8$; $c = 100,000$

Π	TD 88-90	TV 88-90
$i = 3.5\%$	...	74,917
$i = 7\%$	56,412	57,416

▷ interest rate (8 years) more impacting than mortality risk

▷ most prudent pricing: $i = 3.5\%$ & TV (i.e. **regulatory** table)

▷ Cost of the contract (for the insurer) from $t = 0$:

$$X_i = \begin{cases} c.v^k & \text{w/ probability } p = \frac{l_{x+k}}{l_x} \\ 0 & \text{w/ probability}(1-p) \end{cases}$$

▷ thus $\mathbb{E}(X_i) = \Pi$ and $\sigma(X_i) = c.v^k.\sqrt{\frac{l_{x+k}}{l_x}\left(1 - \frac{l_{x+k}}{l_x}\right)}$

- that is for n (identical and independent) contracts: $\mathbb{E}(X) = n.\Pi$ and $\sigma(X) = \sqrt{n}\sigma(X_i)$

- for 10,000 contracts and the prudent pricing above: $\mathbb{E}(X) = 749,170,000$ and $\sigma(X) = 100.\sigma(X_i) = 876.150$

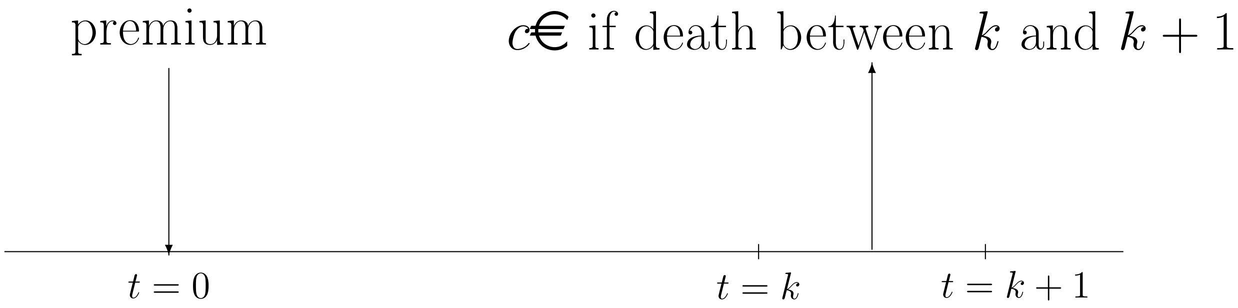
- We thus end up w/ a confidence interval at 95% for the total cost of the (n) contracts:

$$[X]_{95\%} = [747,452,746; 750,887,254]$$

- Relatively small interval $\rightarrow$ **few risk** for the insurer

(Deferred) Term life insurance

- Commitment (at $t = 0$): pay c€ to the beneficiary at the death of the insured IF it occurs between $t = k$ and $t = k + 1$
- In french “temporaire déc’es (différée)”



- **Warning**: paid at death not at the end of the contract
- **Assumption**: deceases are uniformly distributed over the year
  - in expectation decease occurs at $k + \frac{1}{2}$
  - then at $t = k + 1$

$$\mathbb{E}(R_1) = \underbrace{\left(\Pi(1+i)^{k+\frac{1}{2}} - cq\right)}_{\text{en } t = k + \frac{1}{2}} .(1+i)^{\frac{1}{2}}$$

  - where q represent the proba of dying between $t = k$ and $t = k+1$

▷ The fair premium then writes

$$\Pi = \frac{c.\frac{l_{x+k}-l_{x+k+1}}{l_x}}{(1+i)^{k+\frac{1}{2}}}$$

▷ Numerical ex.: $x = 40$, $k = 0$ (immediate), $c = 100,000$

Π	TD 88-90	TV 88-90
$i = 3.5\%$	280	122
$i = 7\%$	275	...

▷ small impact of the i.r. (immediate), huge mortality risk

▷ prudent pricing: $i = 3.5\%$ & TD (i.e. **regulatory** table)

▷ $\sigma(X_i) = c.v^{\frac{1}{2}}\sqrt{\left(1-\frac{l_{x+1}}{l_x}\right)\frac{l_{x+1}}{l_x}} = 5,239.7$ (high)

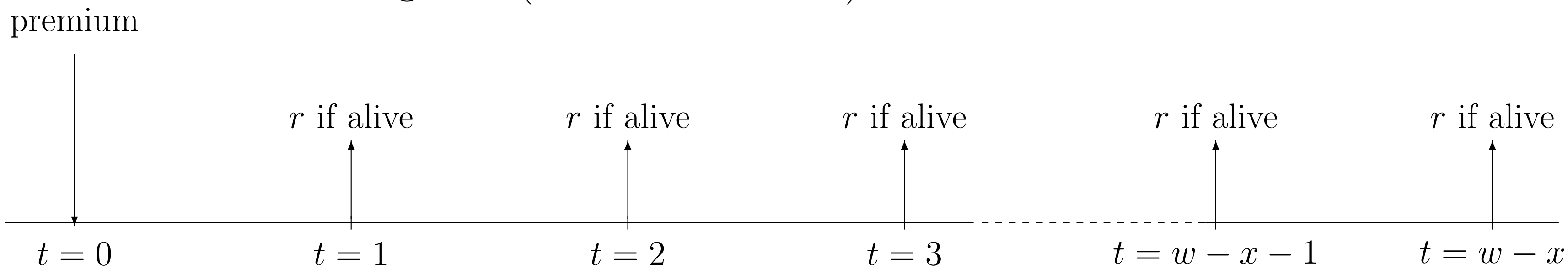
⇒ For 10,000 contracts (with prudent pricing)

$$[X]_{95\%} = [1,774,160; 3,828,120]$$

⇒ **big uncertainty!**

Life annuity (in arrears)

- ▷ Engagement: pay r€ at the end of each year (in arrears) as long as the insured is alive
- ▷ in French: “rente viagàre (à terme échu)”



- ▷ Fair premium:

$$\Pi = \ldots$$

- ▷ Numerical ex.: $r = 10,000$; $x = 65$ (**retirement**)

Π	TD 88-90	TV 88-90
$i = 3.5\%$	107,932	132,524
$i = 7\%$	...	97,581

(amount to be paid at 65 to get 10,000€ a year until death)

- ▷ high impact of both interest rate **and** mortality rate!
- ▷ prudent pricing: $i = 3,5\%$ and TV

▷ Cost of a policy:

$$X_i = \begin{cases} 0 & \text{with probability} \frac{l_x - l_{x+1}}{l_x} \\ \ldots & \text{with probability...} \\ \ldots & \text{with probability...} \\ . & \\ . & \\ . & \\ \ldots & \text{with probability...} \end{cases}$$

▷ hence, using prudent pricing:

$$\mathbb{E}(X_i) = \Pi = 132,524 \text{ and } \sigma(X_i) = \sqrt{\mathbb{E}(X_i^2) - [\mathbb{E}(X_i)]^2} = 44,448.72$$

▷ and for 10,000 policies

$$[X]_{95\%} = [1,316,528,050; 1,333,951,950]$$

Actuarial Present Values and notations

- $T_x \equiv$ random survival time of an individual aged x
- $\mathbb{P}(T_x > k) = \frac{l_{x+k}}{l_x} \equiv {}_k p_x$
- $\mathbb{P}(k < T_x < k + k') = \frac{l_{x+k} - l_{x+k+k'}}{l_x} \equiv {}_{k|k'} q_x$
- Actuarial Present Value of **“pure” products**

$$ {}_{k|k'}\text{APV}_x $$

where k represents the deferred period
and k' the duration

Pure products

▷ **In case of life**: Pure endowment
(1 € paid in k year if the insured aged x is still alive)

$$ {}_kE_x \equiv v^k . \frac{l_{x+k}}{l_x} $$

▷ **In case of death**: Deferred One Year Term
(1 € paid if the insured aged x dies between age $x+k$ and $x+k+1$)

$$ {}_{k|1}A_x \equiv v^{k+\frac{1}{2}} . \frac{l_{x+k} - l_{x+k+1}}{l_x} $$

▷ Whole-life annuity ("rente viagère")

- in advance: $\ddot{a}_x = {}_0E_x + {}_1E_x + ... + {}_{w-x-1}E_x$
- in arrears: $a_x = {}_1E_x + {}_2E_x + ... + {}_{w-x}E_x$

▷ Whole-life term insurance ("Garantie décès vie entiére")
≈ funeral contract ("contrat obsèques")

$$ A_x = {}_{0|1}A_x + {}_{1|1}A_x + ... + {}_{k|1}A_x + ... + {}_{w-x-1|1}A_x $$

Commutation functions

- ▷ To simplify the calculation: commutations functions
- ▷ ∃ tables of commutation functions: for given i.r. and life table
- ▷ **Life commutation functions**

$$D_x \equiv v^x l_x \quad \text{and} \quad N_x \equiv D_x + D_{x+1} + \ldots + D_w$$

give

$$_kE_x = \frac{D_{x+k}}{D_x}, \quad \ddot{\mathrm{a}}_x = \frac{N_x}{D_x}, \quad \mathrm{a}_x = \frac{N_{x+1}}{D_x} \quad \text{and} \quad _{m|n}\ddot{\mathrm{a}}_x = \frac{N_{x+m} - N_{x+m+n}}{D_x}$$

- ▷ **Decease commutation functions**

$$C_x \equiv v^{x+\frac{1}{2}}(l_x - l_{x+1}) \quad \text{and} \quad M_x \equiv C_x + C_{x+1} + \ldots + C_{w-1}$$

give

$$_{k|1}A_x = \frac{C_{x+k}}{D_x}, \quad \mathrm{A}_x = \frac{M_x}{D_x} \quad \text{and} \quad _{m|n}\mathrm{A}_x = \frac{M_{x+m} - M_{x+m+n}}{D_x}$$

Exercises

- A benefit $C = 10,000$€ will be paid to a beneficiary in the event of death in the next 3 years of an individual who is simultaneously the owner and the insured, and who is today aged 50.

  Price (with $i = 3\%$) this policy (i) with a single premium and (ii) with constant annual premiums paid in advance during three years.

- A loan of $K = 10,000$€ is repaid with three constant annual payments of 4,000€ (in arrears). An insurance contract guarantees, in the event of death of the borrower, the repayment of the remaining installment at the due term.

  What is the Actuarial Present Value of this insurance at the time the loan is granted? Do the numerical exercise for an insured aged 40 with an interest rate of 3%.

  Compute the fair constant annual premium to be paid in advance during the life of the loan.

Extensions

▷ Benefit on the first death of (x) and (y)

$$1 - \frac{l_{x+k}}{l_x}.\frac{l_{y+k}}{l_y}$$

▷ Reversible (or joint) life annuity

$$\frac{l_{x+k}}{l_x} + \alpha.\left(1 - \frac{l_{x+k}}{l_x}\right).\frac{l_{y+k}}{l_y}$$

▷ Varying annuities

- ▶ geometric progression $\left((1+\rho)^k\right)$
- ▶ arithmetic progression $(k+1)$

▷ Variable interest rates

$$v_k = \frac{1}{1+i_1}.\frac{1}{1+i_2}...\frac{1}{1+i_k}$$

Non-life insurance (IARD)

Differences with life insurance

- ▷ Shorter term
- ▷ More variability

But also

- ▷ Claim settlement process (slower)

⇒ More complicated accounting (**reserving**)

⇒ Importance of **safety margin** (implicit/regulated in life insurance)

⇒ Importance of investment on the stock **market**

CENTRALE MARSEILLE

amse aix marseille school of economics

ACCOUNTING SPECIFICITIES

▷ Accounting tracks the **amount** of claims not the number!

⇒ Difficult to track the frequency and the average costs of claims

⇒ Profitability measured by the ratio **C/P**:
(sum of) claims to (sum of) premiums ratio ("sinsitres sur prime")

▷ Time for Claim settlement

⇒ differences between the accounting year and the **claim year**

- Incurred But Not (yet) Reported **IBNR** claims
- Reported But Not (yet) Settled **RBNS** claims
- called "tardifs" in French

⇒ In France: three accounting statements

- C1 reflects the accounting year
- C10 and C11 reflects the occurrence year
  (resp. for "claims" and "premiums and profits")

Reserving ("Provisionnement")

- ▷ To ultimately pay the IBNR (& RBNS) claims
- ▷ insurance companies have to set (claim) **reserves** ("PSAP: Provision pour Sinistres À Payer" in French)
- ▷ i.e. to hold liquidity at year n
- ▷ for claims (on contracts) from previous years

- ▷ the difference between reserves and the (real) costs of claims (from $n-k$ in n)
- ▷ determine a **boni** (or a **mali**) from claims reserving ("de liquidation de provisionnement" in French)

A simple example

▷ Consider next example where we want

▷ to study the **changes in reserving** in the end of year n

▷ to determine boni and mali

		$n-4$	$n-3$	$n-2$	$n-1$	n	Total
	Settlement at year n	1	2	10	177	294	484
+	Reserves on $12/31/n$	1	2	4	15	140	162
-	Reserves on $01/01/n$	2	5	14	187		208
							438
=	Costs of claims incurred in n					434	434
+	Costs of claims incurred before n	0	$\underbrace{-1}_{\text{boni}}$	0	$\underbrace{5}_{\text{mali}}$		4

▷ Claims reserving led to a **mali** of 4

▷ because of **misevaluated** reserves/provisions for year $n-1$

Evaluating claim reserves: the Chain-Ladder method

- ▷ How to make (and update) forecast on outstanding claims (incl. IBNR & RBNS)?
- ▷ Most popular method: Chain-Ladder
- ▷ **Assumption**: ∃ a regularity in the **cadence of payments**
- ▷ Use of incremental payments $X_{i,j}$
  i.e. the payments made in $i+j$ for claims incurred in i
- ▷ and cumulative payments $C_{i,j} = X_{i,0} + X_{i,1} + ... + X_{i,j}$
- ▷ The Chain-Ladder method assumes

$$C_{i,j+1} = \lambda_j . C_{i,j}, \qquad \forall i, j$$

i.e. ∃ a **recurrence relation** on cumulative payments

Cumulative payments and reserving

▷ After t year, the amount remaining to be paid for claims of year i writes

$$R_{i,t} = C_{i,\infty} - C_{i,t}$$

▷ And the **reserves** (provision) will correspond to the **forecast**

$$\widehat{R}_{i,t} = \mathbb{E}\left(R_{i,t} \mid \mathcal{F}_t\right) = \mathbb{E}\left(C_{i,\infty} \mid \mathcal{F}_t\right) - C_{i,t}$$

where $\mathcal{F}_t$ represents the information available after t years

$$\mathcal{F}_t = \left\{(C_{i,j}, 0 \leq i+j \leq t\right\} = \left\{(X_{i,j}, 0 \leq i+j \leq t\right\}$$

▷ *Remark* : $\left(\mathbb{E}\left(C_{i,\infty} \mid \mathcal{F}_t\right)\right)_t$ is a martingale

▷ the Chain-Ladder method consist in estimating the λ_js

▷ on the basis of observations on n years ($n-j$ obs for each j)

The Chain-Ladder estimate

▷ Chain-Ladder estimate: weighted **average ratio** on the $n-j$ obs.

$$\widehat{\lambda}_j = \frac{\sum_{i=1}^{n-j-1} C_{i,j+1}}{\sum_{i=1}^{n-j-1} C_{i,j}}$$

▷ i.e. $\widehat{\lambda}_j = \sum_{i=1}^{n-j-1} \omega_{i,j}.\lambda_{i,j}$ with $\omega_{i,j} \equiv \frac{C_{i,j}}{\sum_{i=1}^{n-j-1} C_{i,j}}$ and $\lambda_{i,j} = \frac{C_{i,j+1}}{C_{i,j}}$

▷ *Remark* : $\widehat{\lambda}_j = \underset{\lambda\in\mathbb{R}}{\arg\min}\left\{\sum_{i=1}^{n-j} C_{i,j}.\left[\lambda - \frac{C_{i,j+1}}{C_{i,j}}\right]^2\right\}$

(can come from a weighted least-square linear reg. without cst of $C_{i,j+1}$ on $C_{i,j}$)

▷ We can then estimate the cumulative payments

$$\widehat{C}_{i,j} = \left[\widehat{\lambda}_{n-i+1}\cdots\widehat{\lambda}_{j-1}\right] C_{i,n-i+1}$$

▷ and the claim reserves
(assuming that all the claims have been settled after n year)

Example (1)

$X_{i,j}$	0	1	2	3	4	5
1	3209	1163	39	17	7	21
2	3367	1292	37	24	10	
3	3871	1474	53	22		
4	4239	1678	103			
5	4929	1865				
6	5217					

$C_{i,j}$	0	1	2	3	4	5
1	3209	4372	4411	4428	4435	4456
2	3367	4659	4696	4720	4730	
3	3871	5345	5398	5420		
4	4239	5917	6020			
5	4929	6794				
6	5217					

▷ We then have

$$\widehat{\lambda}_0 = 1.38093 \; ; \; \widehat{\lambda}_1 = 1.01143 \; ; \; \widehat{\lambda}_2 = 1.00434 \; ; \; \widehat{\lambda}_3 = \ldots \; ; \; \widehat{\lambda}_4 = 1.00474$$

▷ and we can complete the table

$C_{i,j}$	0	1	2	3	4	5
1	3209	4372	4411	4428	4435	4456
2	3367	4659	4696	4720	4730	**4752.4**
3	3871	5345	5398	5420	**5430.1**	**5455.8**
4	4239	5917	6020	...	**6057.4**	**6086.1**
5	4929	6794	**6871.7**	**6901.5**	**6914.3**	**6947.1**
6	5217	**7204.3**	**7286.7**	**7318.3**	**7331.9**	**7366.7**

Example (2)

- ▷ Assuming that 5 years are enough to settle all the claims
- ▷ the insurer has to set up reserves up to
  - ▶ 22.4 for year 2
  - ▶ 35.8 for year 3
  - ▶ 66.1 for year 4
  - ▶ ... for year 5
  - ▶ 2149.7 for year 6
- ▷ That is a total of 2427.1

- ▷ The **year after**, we observe an additional diagonal,
- ▷ what changes the estimations and therefore the reserves
- ▷ creating **boni and mali**

Extensions

▷ Probabilistic models (also use $\text{Var}(C_{i,j+1} \mid C_{i,j})$)

$$C_{i,j+1} = \lambda_j . C_{i,j} + \sigma_j . \sqrt{C_{i,j}} . \epsilon_{i,j}$$

with $\widehat{\sigma}_j^2 = \frac{1}{n-j-1} \sum_{i=1}^{n-j-1} \left(\frac{C_{i,j+1}}{C_{i,j}} - \widehat{\lambda}_j \right)^2 . C_{i,j}$

▷ Econometric models (Poissonian regression)

- Assumptions
  - a year effect, and a delay effect
  - multiplicative effect
- $X_{i,j} \sim \mathcal{P}(A_i . B_j) \Rightarrow \mathbb{E}(X_{i,j}) = A_i . B_j$
- $\widehat{X}_{i,j} = \widehat{A}_i . \widehat{B}_j$
- provides the same forecast as the Chain-Ladder estimate

Fair premium and Non-life risk

- Contrary to a life insurance contract
- **several claims** can occur on a single non-life contract
- The cost of a policy (X) the depends on:
  - the number of claims on this policy: N (random)
  - the cost of each of these claims: Y_i, $i = 1, ..., N$ (random)

  with $X = Y_1 + ... + Y_N$
- the **fair premium** will then be:

$$\Pi = \mathbb{E}(X) = \mathbb{E}_N[\mathbb{E}(X \mid N)] = \mathbb{E}_N[\mathbb{E}(Y_1 + ... + Y_N \mid N)]$$

- Then, if
  - the Y_{ij}s (the costs of the j^{th} claim of individual i) are i.i.d. knowing N_i (the # of claims of ind. i)
  - the N_is are i.i.d.

  $\mathbb{E}(X) = \mathbb{E}_N[\mathbb{E}(N.Y)] = \mathbb{E}(N).\mathbb{E}(Y)$

Variability of a non-life risk

▷ Similarly, the variance of this cost depends on both

- the **variability in the number** of claims per contract
- the **variability in the cost** of a claim

▷ Under the above assumptions:

$$\begin{aligned}
\mathbb{E}(X^2) &= \mathbb{E}_N[\mathbb{E}(X^2 \mid N)] = \mathbb{E}_N[\mathbb{E}((Y_1 + ... + Y_N)^2 \mid N)] \\
&= \mathbb{E}_N\left[\mathbb{E}\left(\textstyle\sum_{i=1}^N Y_i^2 \mid N\right) + \textstyle\sum_{i=1}^N \sum_{j\neq i} \mathbb{E}(Y_iY_j \mid N)\right] \\
&= \mathbb{E}_N\left[N.\mathbb{E}(Y^2) + N.(N-1)\mathbb{E}(Y)\mathbb{E}(Y)\right] = \mathbb{E}(N).\mathrm{Var}(Y) + \mathbb{E}(N^2).\left[\mathbb{E}(Y)\right]^2 \\
\text{and} & \\
\mathrm{Var}(X) &= \mathbb{E}(N).\mathrm{Var}(Y) + \mathbb{E}(N^2).\left[\mathbb{E}(Y)\right]^2 - \left[\mathbb{E}(N).\mathbb{E}(Y)\right]^2 \\
&= \left[\mathbb{E}(Y)\right]^2.\mathrm{Var}(N) + \mathbb{E}(N).\mathrm{Var}(Y)
\end{aligned}$$

▷ And in the **particular case** where $\mathrm{Var}(N) = \mathbb{E}(N)$ (for ex. if $N \sim \mathcal{P}$)

$$\mathrm{Var}(X) = \mathbb{E}(N).\mathbb{E}(Y^2)$$

Example

Consider a portfolio of 400,000 identical contracts for which

- ▷ the number of claims per contract can be approximated by a $\mathcal{P}(0.07)$
- ▷ the claims lower than $M = 200,000$ € have an expectation $C_1 = 10,540$ € and a standard error $\sigma_1 = 19,000$ €
- ▷ a proportion $p = 1\%$ of claims are higher than M (for clipping purpose, "écrétage"). The expectation of these big claims is $C_2 = 410,000$ € and their standard error $\sigma_2 = 1.3$ M€
- ▷ the number of claims per contract are assumed to be i.i.d., and given these numbers, the size of claims are also assumed to be i.i.d.

  - ▶ Compute the annual fair premium (on a contract)
  - ▶ Compute the standard error of the cost of a contract
  - ▶ The insurer evaluates its charges to 15% of the commercial premium Π''. Compute the value of Π'' that makes lower than 10% the prob that the insurer looses – on its entire portfolio – **exceed** 20 M€

Link with the regulation

- ▷ the **Value at risk** at $1-\alpha\%$ ($\text{V@R}_{1-\alpha}$): the potential loss than can occur on a portfolio with a proba α
- ▷ Quantile of level α of the distrib of profits and losses X: $\mathbb{P}(X > \text{V@R}_{1-\alpha}) = \alpha$

- ▷ **Solvency 2** : the Solvency Capital Requirement (SCR)
  - ▶ **target** level of own funds the insurance company should aim for
  - ▶ corresponds to a **Value at risk** at 99.5% over one year
  - ▶ capital that enables the insurer to absorb bicentennial (adverse) events

Non-life insurance and financial markets

- In life insurance: use of risk-free interest rate i
- In non-life, no assumption on the investment of premiums income nor on the investment of **reserves**
- whereas, it has an direct impact on insurer's profit
- Even in the case of a decrease in loss ratio,
- the financial equilibrium can be threatened by "bad" investments that is a degradation of assets
- Case study: the evolution of car insurance price (by Gilbert THIRY, Consultant)

Case study – Calculation assumptions (1)

▷ Financial equilibrium obtained for a claims-to-premiums ratio C/P=78%

▷ Technical result

Asset	Liability
Premiums: 100	Claims: 78
Financial products: 7	Overhead costs: 29

▷ After one year

- ▶ Annual claims frequency: -6%
- ▶ Average cost: +2%

⇒ Cost of claims: -4% (0.94x1.02=0.96)

- ▶ Financial products: -10%
- ▶ Overhead costs: +2%

Case study – Calculation assumptions (2)

- ▷ The same technical result can then be obtained
- ▷ by decreasing premiums by 1.8%

Asset	Liability
Premiums: ...	Claims: 74.9
Financial products: 6.3	Overhead costs: 29.6

- ▷ **Issue**: the fall of financial markets also led to
- ▷ a loss on the **investment of reserves**
- ▷ that represent 1.2 times the annual premiums

- ▷ For the average structure of investment by insurance companies
- ▷ a fall of 30% on the shares portfolio gives:

	$y-1$	y
Bonds	66	66
Shares	25	17.5
Real estate and other investments	9	9
Total	100	92.5

Case study – Increase in premiums

- To reconstruct reserves
- the insurance companies should then increase premiums by:

$$7.5\% \text{ x } 1.2 = 9\%$$

- This increase is mitigated by good technical results
- so to avoid losses,
- premiums has to increase by

$$1.09 \text{ x } (1 - 0.982) = 1.07 \;\; \text{that is } 7\%$$

Case study – Exercise

- This result is obviously impacted by the portfolio structure
- Under the same assumptions, the increase in premiums needed for two companies

	Company A	Company B
Bonds	51	81
Shares	40	10
Real estate and others	9	9

- will be highly impacted by the proportion of shares in the investment